CLOSED EQUATIONS FOR THE MOTION OF A LIQUID CONTAINING BUBBLES (SURVEY)

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Papers concerned with the equations of motion of multiphase media can be divided conditionally into two categories depending on the approach used: phenomenological or kinetic. In the present paper we adopt the second approach [1]; we obtain closed (i.e., not containing empirical constants and functions) equations, in the simplest case, for the motion of a mixture of a liquid with bubbles of a gas or a vapor: 1) the liquid is nonviscous and incompressible; 2) the velocity field of the liquid exterior to the bubbles is irrotational; 3) the bubbles have a spherical shape and the pressure in a bubble is a given function of its radius; 4) the volume concentration of bubbles and the mean distance between neighboring bubbles is small. These equations can be a basis of phenomenological theories, which take into account the properties of real mixtures more completely.

We discuss in detail known closed models of bubble-containing media [2-8]. We prove convergence for the method of successive approximations for the calculation of the liquid velocity potential exterior to the bubbles. The equations of motion for a system of spherical particles, obtained in [9-11], are generalized to the case of pulsating bubbles. In the framework of the resulting hydrodynamic equations we calculate the rate of propagation of small perturbations. We solve the one-dimensional problem concerning the decomposition of an initial discontinuity in which, on one side of the discontinuity surface, there is a pure liquid with a constant pressure.

The phenomenological approach was developed in [12-14]; these references also contain an appropriate bibliography.

1. Consideration of Known Models

The equations for the motion of a suspension of bubbles of a gas or a vapor in a liquid can be divided into two categories: 1) phenomenological equations containing, in addition to the characteristics of the pure phases, the constants or functions for the mixture, which must be determined experimentally; 2) closed equations, which employ only the characteristics of the mixture components taken individually. We shall consider equations of the second type. These equations are valid when the volume concentration of the gaseous phase of the mixture is small. In order for the mixture to be regarded as a continuous medium the mean distance l between neighboring bubbles must be small in comparison with a characteristic flow dimension M.

Let $\mathbf{u}_0(\mathbf{x}, t)$ and $p_0(\mathbf{x}, t)$ denote the velocity and pressure of the liquid at the midpoint between bubbles; at other points we define these functions by some method of interpolation. Derivation of the equations of motion is based on two assumptions. First of all, we assume that $\mathbf{u}_0$ and p_0 are regular functions of the spatial point $\mathbf{x} = (x^1, x^2, x^3)$ and of the time t. We assume the function $\Phi(\mathbf{x}, t)$ to be regular if

$$|\nabla \Phi| l + |\partial\Phi / \partial t| l / v \ll |\Phi|$$

where v is the mean speed of the bubbles and

$$\nabla = (\partial / \partial x^1,\ \partial / \partial x^2,\ \partial / \partial x^3)$$

This assumption is valid for small volume concentrations of the bubbles. Secondly, we assume the individual trajectory of the center of each bubble, namely $\mathbf{x}_i(t)$, to be a regular function of the time t. In

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other words, the speed of a bubble is a regular function of its coordinates: $\mathbf{v}_i(t) = \mathbf{v}(\mathbf{x}_i, t)$. At points $\mathbf{x} \neq \mathbf{x}_i$ we define the function $\mathbf{v}(\mathbf{x}, t)$ by interpolation. This property is required to take into account the appearance of compressibility of the bubbles. Bubble interaction through pulsations has the character of a long-range action; we can therefore neglect collisions of a given bubble with neighboring bubbles, as a result of which the trajectory of the given bubble undergoes a break, and take into account only the smooth collective actions of the distant bubbles. This property does not hold for a suspension of incompressible particles in a liquid, wherein the particle interaction is of a short-range nature.

We describe exactly a class of flows for which we obtain the equations of motion. We indicate a typical representative of this class:

The Borzhom Problem. In an unbounded ideal and incompressible liquid let there be N cavitational bubbles of spherical shape and of the same radius r. These bubbles are uniformly distributed in a ball of diameter M, the distance between bubbles is l, and the volume concentration of the bubbles is c. The pressure p^+ is the same in all the bubbles and is constant in time. The pressure is equal to zero at infinity. At the initial time instant the liquid velocity field is equal to zero. This is approximately the situation (for $p^+ > 0$) when a Borzhom bottle is opened. We assume that as they move the bubbles retain their spherical shape, the volume concentration of the bubbles $c \ll 1$, and $l \ll M$. We wish to describe this motion in a first approximation with respect to c and with respect to l. Of interest is the limiting case $l = 0$ for $c > 0$.

When $c = 0$, we have quiescence; therefore the presence of the bubbles is the principal effect in this problem and manifests itself in a first approximation with respect to c. The formulated Borzhom problem can serve as a special test problem for verifying various models of bubble-containing media.

We make the following assumptions with reference to the two-phase flows considered.

1) The liquid is nonviscous, incompressible, with density equal to one; it is unbounded, and the external force fields have the potential U.

2) The liquid motion is irrotational and the speed and pressure of the liquid at infinity are equal to zero.

3) The bubbles have a spherical shape. The pressure in a bubble is a given function $p^+(r)$ of its radius and is the same for all the bubbles. The special case $p^+ = \text{const}$ is admissible. The density of the gas in a bubble is taken equal to zero since it is many times less than the density of a liquid drop.

4) The mean radius r of the bubbles and the mean distance l between neighboring bubbles satisfy the inequalities

$$r \ll l \ll M$$

where M is a characteristic dimension of the average flow.

We note that all the assumptions are satisfied in the Borzhom problem.

The absence of rigid boundaries in 1) is essential. In the condition 3) p^+ is the pressure on the outside of the bubble wall; this pressure differs from the pressure inside the bubble by the presence of surface tension. By virtue of the conditions 2) and 3) the velocity field of the liquid is determined uniquely by the coordinates $\mathbf{x}_i$ of the centers and the radii r_i of the bubbles and also by their rates of change

$$\mathbf{v}_i = \frac{d\mathbf{x}_i}{dt}, \quad s_i = \frac{dr_i}{dt} \quad (i = 1, \ldots, N)$$

Consequently, the motion of the system consisting of the liquid and the bubbles is determined by a finite number (8N) of scalar parameters. Therefore the evolution of the system in time is described by a Lagrange system of ordinary differential equations of order 8N.

Assumption 4) concerning the low gas concentration makes it possible to write this system of equations in explicit form. It is evident from this assumption that the force of interaction of two bubbles decreases as d^{-k}, $k \leq 2$, as the distance d between the bubbles increases. For $k \leq 3$ it is customary in the kinetic theory of gases to refer to the particle interaction forces as long-range forces. Long-range action between the gas bubbles in a liquid arises thanks to pulsation of the bubbles. This circumstance makes it possible to let $N \to \infty$ in the system of Lagrange equations and to obtain the hydrodynamic equations (Section 4). In Section 6 we solve a one-dimensional version of the Borzhom problem.

This method of describing the motion of bubble-containing media and, correspondingly, the new equations of motion was proposed by the author in [1]. It is appropriate to compare these equations with the equations of other authors. The latter are considered in connection with flows satisfying the conditions 1)-4). We recall that phenomenological equations of motion of bubble-containing media, which involve empirical constants or functions, are not considered here.

Quasistatic Model [2, 3]. In this model the velocity and pressure fields, $\mathbf{u}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$, are assumed to be regular outside the bubbles:

$$\mathbf{u} = \mathbf{u}_0 = \mathbf{v} \tag{1.1}$$

$$p = p_0 = p^+(r) \tag{1.2}$$

Let N be the number of bubbles per unit mass of the mixture. Then the density of the mixture is

$$\langle \rho \rangle = (1 + {}^4/_3 \pi r^3 N)^{-1} \tag{1.3}$$

Thanks to the assumption (1.1) the continuity equation and the momentum equation coincide with the equations for a gas. Equations (1.2) and (1.3) yield the equation of state

$$p = p^+ ((\langle \rho \rangle^{-1} - 1)^{1/3} ({}^4/_3 \pi N)^{-1/3}) \tag{1.4}$$

The quantity N plays the role of entropy since it is conserved along the trajectories of the bubbles (of the liquid particles).

S. V. Iordanskii's Equations [4]. We consider in detail the method proposed in [4]. Let $\mathbf{u}$ and p be the velocity and pressure in the liquid, functions which are defined outside of the bubbles. We define the pressure inside the bubble Ω_i to be equal to $p^+(r_i)$. We define the function ρ to be equal to 1 and 0 in the liquid and in the bubbles, respectively. The functions ρ, $\mathbf{u}$, and p are nonregular (they exhibit strong variation at distances of the order l); therefore their averaged values are of interest for experimental measurements. We define the averaging operation by

$$\langle F(\mathbf{x}_0, t) \rangle = L^{-3} \int_V F(\mathbf{x}, t)\, d\mathbf{x}$$

where V is a cube with center at the point $\mathbf{x}_0$ and with edge L, $l \ll L \ll M$. We apply the averaging operation to the equation of motion of a nonviscous incompressible liquid. Since the operations of averaging and differentiation are commutative, we obtain

$$\begin{gathered} \frac{\partial}{\partial t} \langle \rho \rangle + \frac{\partial}{\partial x^\beta} \langle \rho u^\beta \rangle = 0 \\ \frac{\partial}{\partial t} \langle \rho u^\alpha \rangle + \frac{\partial}{\partial x^\beta} (\langle \rho u^\alpha u^\beta \rangle + \langle p \rangle \delta_{\alpha\beta}) = 0 \qquad (\alpha = 1, 2, 3) \end{gathered} \tag{1.5}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol and repeated indices denote summation from 1 to 3.

We define regular functions c, $\mathbf{u}_0$, p_0, $\mathbf{v}$, r, and s, in terms of which we shall describe the averaged motion of the mixture and for which we need to obtain a closed system of equations. We express $\langle \rho \rangle$, $\langle \rho u^\alpha \rangle$, $\langle \rho u^\alpha u^\beta \rangle$, $\langle p \rangle$ in terms of these functions. To do this we assume the following flow structure in the averaging region V:

1) all the $\mathbf{v}_i$, r_i, and s_i are identical in V and are equal, respectively, to

$$\mathbf{v}_1(t) = \mathbf{v}(\mathbf{x}_1, t),\ r_1(t) = r(\mathbf{x}_1, t),\ s_1(t) = s(\mathbf{x}_1, t)$$

the number of bubbles in V is given by

$$nL^3 = c(\mathbf{x}_0, t)({}^4/_3 \pi r_1^3)^{-1} L^3$$

where $\mathbf{x}_0$ is the center of the cube V; $\mathbf{x}_1$ is the center of some bubble in V;

2) we surround each bubble in V by a sphere S of maximum radius R, so that $R \sim l/2$. Outside of these spheres and within the confines of V we consider the velocity and pressure in the liquid to be constant and equal to

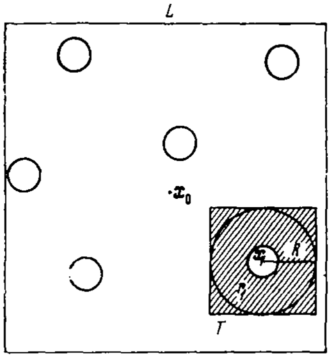

Fig. 1

$$\mathbf{u}_0(t) = \mathbf{u}_0(\mathbf{x}_0, t), \; p_0(t) = p_0(\mathbf{x}_0, t)$$

3) in the interior of each region Q between the sphere S and the bubble surface we consider the velocity and pressure fields to be the same as those for the motion of an isolated bubble in the unbounded liquid with speed $u_0(t)$ and pressure $p_0(t)$ at infinity.

We obtain

$$\langle \rho \rangle = 1 - c$$

$$\langle \rho \mathbf{u} \rangle = \mathbf{u}_0 (1 - n{}^4/{}_3 \pi R^3) + n \int_Q \mathbf{u}\, dx = (1 - c)\, \mathbf{u}_0 + \int_Q \nabla \varphi \, dx \tag{1.6}$$

The velocity field of the liquid during the motion of an isolated bubble has the potential $\mathbf{u} - \mathbf{u}_0 = \nabla \varphi$, which we determine explicitly in Section 2. The integral over Q in this equation is equal to zero; therefore

$$^0\mathbf{n}\langle d \rangle = \langle \mathbf{n} d \rangle \tag{1.7}$$

If instead of Q we take the region T (the shaded region in Fig. 1, in which the averaging region is indicated), the integral in Eq. (1.6) is then not equal to zero and we obtain

$$\langle \rho \mathbf{u} \rangle = \langle \rho \rangle \mathbf{u}_0 + kc\mathbf{w} \tag{1.8}$$

where k is some number and $\mathbf{w} = \mathbf{v} - \mathbf{u}_0$.

Similarly we obtain

$$\begin{aligned} \langle \rho u^\alpha u^\beta \rangle &= \langle \rho \rangle u_0^\alpha u_0^\beta + c\,((s^2 + {}^3/{}_{20} |w|^2)\, \delta_{\alpha\beta} + {}^1/{}_{20} w^\alpha w^\beta) + O(c^{4/3}) \\ \langle p \rangle &= p_0 + c\,({}^3/{}_2 ((R / r)^2 - 1)\, r^{-1} d(r^2 s) / dt + p^+(r) - p_0 - {}^3/{}_2 s^2 - {}^1/{}_4 |w|^2) + O(c^{4/3}) \end{aligned} \tag{1.9}$$

From assumption 3) we obtain equations describing the motion of an individual bubble:

$$\frac{d}{dt}(r^3 \mathbf{w}) = 2r^3 \left(\frac{\partial \mathbf{u}_0}{\partial t} + (\mathbf{u}_0 \cdot \nabla)\, \mathbf{u}_0 \right) - r^3 (\mathbf{w} \cdot \nabla)\, \mathbf{u}_0 \tag{1.10}$$

$$r\, ds/dt + {}^3/{}_2 s^2 = {}^1/{}_4 |\mathbf{w}|^2 + p^+(r) - p_0 \tag{1.11}$$

$$dr/dt = s \qquad (d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla) \tag{1.12}$$

where the dot indicates the scalar product of the vectors and $|\mathbf{w}|$ denotes the length of the vector $\mathbf{w}$.

The conservation equation for the number of bubbles has the form

$$\partial n / \partial t + \partial (n v^\beta) / \partial x^\beta = 0 \tag{1.13}$$

where n is the number of bubbles per unit volume of the mixture:

$$n = {}^3/{}_4 \pi^{-1} r^{-3} c \tag{1.14}$$

The equations (1.5)-(1.7) and (1.9)-(1.14) form a closed system. The quantity R, whose exact value cannot be determined theoretically in the scheme set forth here, enters the equation in the lowest degreee with respect to r and c:

$$(R / r)^2 \approx (l / r)^2 \approx K c^{-2/3}$$

where the constant K can only be determined experimentally. In [5] it was assumed that

$$\langle p \rangle \approx p_0 \tag{1.15}$$

This is inadmissible if the accepted accuracy is to be maintained.

B. S. Kogarko's Model [5]. We obtain the equations given in [5] if in the system of equations given in [4] we put $\mathbf{v} = \mathbf{u}_0$, discard the equation (1.10) contradictory to this assumption, and adopt, instead of the relation (1.9), the assumptions (1.15) and

$$\langle \rho u^\alpha u^\beta \rangle \approx \langle \rho \rangle u_0^\alpha u_0^\beta \tag{1.16}$$

The equations given in [5] have the usual gasdynamic form but use, instead of an equation of state, an equation for the radial pulsation of a bubble. We write the pulsation equation, taking into account the viscosity of the liquid:

$$r\frac{d^2r}{dt^2} + {}^3/_2\left(\frac{dr}{dt}\right)^2 + 4\nu r^{-1}\frac{dr}{dt} = p^+(r) - p_0 \tag{1.17}$$

where ν is the coefficient of kinematic viscosity of the liquid.

Analogous equations appeared later on in [6-8] and were generalized to include viscosity and thermal conductivity of the liquid.

In the models indicated above yet additional assumptions were used besides the assumptions 1)-4). We shall use here only the assumptions 1)-4). In this there is a basic difference between our method and those indicated above. The equations given in [4, 5] and the equations we give in Section 5 coincide in the case of small oscillations in the neighborhood of quiescence. Is it possible to obtain in the limit for $l \to 0$ and $c > 0$ (for example, in the stopper problem) a quasistatic model? This cannot be the case since $\mathbf{v} \not\to \mathbf{u}_0$. The equations given in [5] yield a quasistatic model in the limit. The equations given in [4] do not go over in the limit to the quasistatic equations; however, they do not even coincide with the limiting equations given in this paper. The reason for this is that, instead of the relation (1.7), it is necessary to take the more general equation (1.8). The value of the constant k in equation (1.8) cannot be calculated by the method given in [4]. In Section 2 we find k by calculation to be equal to $1/2$.

2. Calculation of the Velocity Potential of the Liquid

The velocity potential $\varphi(\mathbf{x}, t)$ is a harmonic function in the liquid interior Ω, and it satisfies on the bubble surfaces Γ_i the flow-over condition; thus

$$\begin{gathered}\Delta\varphi = 0 \quad \text{in } \Omega, \quad \varphi / _\infty = 0 \\ \partial\varphi / \partial n = s_i + \mathbf{v}_i \cdot \mathbf{n} \quad \text{on } \Gamma_i \quad (i = 1, \ldots, N)\end{gathered} \tag{2.1}$$

where the vector $\mathbf{n}$ is normal to Γ_i and is directed towards the liquid interior.

We seek the velocity potential in the form of a sum

$$\begin{gathered}\varphi = \sum_{i=1}^{N} \varphi_i, \qquad \varphi_i = \Phi_i^0 + \Phi_i \\ \Phi_i^0 = -s_i r_i^2 |x - x_i|^{-1} \\ \Delta\Phi_i = 0 \text{ outside } \Gamma_i, \quad \Phi_i|_\infty = 0, \qquad \frac{\partial\Phi_i}{\partial n} = w_i \quad \text{on } \Gamma_i\end{gathered} \tag{2.2}$$

The potential can be represented by the sum (2.2). To prove this it is sufficient to establish the solvability of the system of integral equations for the N unknown functions w_i (w_i is defined on Γ_i)

$$w_i + \sum_{j\neq i} \frac{\partial \Phi_j}{\partial n}\bigg|_{\Gamma_i} = \mathbf{v}_i \cdot \mathbf{n} - \sum_{j\neq i} \frac{\partial \Phi_j^0}{\partial n}\bigg|_{\Gamma_i} \equiv f_i \tag{2.3}$$

<u>THEOREM 2.1.</u> Let $R_0 > 0$, $0 < C_0 < 1$. Let us assume that

$$\begin{gathered} r_i \geqslant R_0 \quad (i = 1, \ldots, N) \\ r_i + r_j \leqslant C_0 \, |\mathbf{x}_i - \mathbf{x}_j| \quad (i, j = 1, \ldots, N)\end{gathered}$$

all the bubbles being located in the interior of some ball of diameter M.

Then there exists a number C, independent of R_0 and C_0, such that for

$$C_0^3 \ln (MC_0 / R_0) < C$$

the system of integral equations (2.3) has a continuous solution; moreover, this solution is unique for arbitrary s_i and $\mathbf{v}_i$. This solution can be obtained by the method of successive approximations and satisfies the inequality

$$\|w_i\| \leqslant (C - C_0^3 \ln(MC_0 / R_0))^{-1} \max_j \|f_j\| \tag{2.4}$$

where

$$\|w_i\|^2 \equiv r_i^{-2} \int_{\Gamma_i} |w_i|^2 \, dS$$

Proof. Let $C_0 \leq 1/2$. Since $\sum_{j\neq i} \varphi_j$ is a harmonic function inside Γ_i, then, if we integrate the i-th equation in (2.3) over Γ_i, we obtain

$$\int_{\Gamma_i} w_i \, dS = \int_{\Gamma_i} f_i \, dS = 0 \tag{2.5}$$

Subject to this condition, there follows from the Poisson formula for the solution of the Neumann problem outside the ball the existence of a number $C_1 < \infty$ such that the inequalities

$$|\Phi_i| \leqslant C_1 r_i^3 / 2 \; |\mathbf{x} - \mathbf{x}_i|^{-2} \; \|w_i\|$$
$$|\nabla\Phi_i| \leqslant C_1 r_i^3 \; |\mathbf{x} - \mathbf{x}_i|^{-3} \; \|w_i\| \quad \text{for} \quad |\mathbf{x} - \mathbf{x}_i| \geqslant 2r_i \tag{2.6}$$

are satisfied.

We seek a solution of the system (2.3) in the class of functions with the finite norm

$$\|w\| = \max_i \|w_i\|$$

From the relations (2.5) and (2.6) we obtain

$$\max_{\Gamma_i} \left| \sum_{j\neq i} \frac{\partial \Phi_j}{\partial n} \right| \leqslant 8C_1 \sum_{j\neq i} r_j^3 |\mathbf{x}_i - \mathbf{x}_j|^{-3} \|w\|$$

Therefore the norm of the integral operator in the system of equations (2.3) does not exceed

$$C_2 = 8C_1 \max_i \sum_{j\neq i} r_j^3 |\mathbf{x}_i - \mathbf{x}_j|^{-3} \tag{2.7}$$

We find an upper bound for the sum (2.7) for all possible distributions of the bubbles satisfying all the hypotheses of the theorem. The meaning of the second assumption is that if we increase the bubble radii C_0^{-1}-fold, then the bubbles must intersect. Therefore the sum (2.7) will be its largest for a dense packing of balls of radius $C_0^{-1}r_j$. This sum then does not exceed the upper Darboux sum for the integral over the region $r_i C_0^{-1} \leq |\mathbf{x} - \mathbf{x}_i| \leq M$:

$$3/4 \pi^{-1} C_0^3 \int |\mathbf{x} - \mathbf{x}_i|^{-3} \, d\mathbf{x} \leqslant 3C_0^3 \ln(MC_0 / R_0)$$

When $C_0^3 \ln(MC_0/R_0)$ is sufficiently small, the norm of the integral operator (2.3) becomes less than one. The theorem then follows from this.

This completes the proof of the theorem.

The volume concentration of the bubbles $c \leq C_0$, so that C_0 is an upper bound for allowable concentrations. The largest allowable values of c decrease slowly as $M/R_0 \to \infty$; therefore we can apply the expansion (2.2) even to the case $M/R_0 = \infty$.

Let $\{P_{m,\alpha}(\mathbf{x})\}$ be a system of homogeneous harmonic polynomials orthonormal on the sphere $|\mathbf{x}| = 1$, let m be the degree of a polynomial, and let $1 \leq \alpha \leq 2m + 1$. We expand the function w_i in a Fourier series:

$$w_j = \sum_{m=1}^{\infty} \sum_{\alpha=1}^{2m+1} w_i^{m,\alpha} P_{m,\alpha}(\mathbf{n}) \tag{2.8}$$

Here, by virtue of the relations (2.5), the zeroth harmonic is absent. We write Parseval's equality

$$\|w_i\|^2 = \sum_{m=1} |w_i^m|^2, \quad |w_i^m|^2 = \sum_{\alpha=1}^{2m+1} |w_i^{m,\alpha}|^2 \tag{2.9}$$

We readily find that

$$\Phi_i = \sum_{m=1}^{\infty} \Phi_i^m$$

$$\Phi_i^m(\mathbf{x}, t) = -r_i^{m+2}/(m+1) \sum_{\alpha=1}^{2m+1} w_i^{m,\alpha} P_{m,\alpha}(\mathbf{x} - \mathbf{x}_i) |\mathbf{x} - \mathbf{x}_i|^{-2m-1} \tag{2.10}$$

Let us estimate $w_i^{m,\alpha}$.

For this we multiply the i-th of the equations (2.3) by $P_{k,\alpha}(\mathbf{n})$ and integrate over Γ_i. By definition we have

$$w_i^{k,\alpha} = r_i^{-2} \int_{\Gamma_i} w_i P_{k,\alpha}(\mathbf{n})\, dS$$

Let us transform the integral

$$A_{ijm}^{k,\alpha} = r_i^{-2} \int_{\Gamma_i} \partial\Phi_j^m / \partial n P_{k,\alpha}(\mathbf{n})\, dS$$

We expand the harmonic function Φ_j^m in a Taylor series in the ball $|\mathbf{x} - \mathbf{x}_i| \le R_j < |\mathbf{x}_j - \mathbf{x}_i|$:

$$\Phi_j^m(\mathbf{x}, t) = \sum_{k=0}^{\infty} \sum_{|\beta|=k} \frac{1}{\beta!} \nabla^\beta \Phi_j^m(\mathbf{x}_i, t)(\mathbf{x} - \mathbf{x}_i)^\beta \tag{2.11}$$

where $\beta = (\beta_1, \beta_2, \beta_3)$ is a vector index for which

$$|\beta| = \beta_1 + \beta_2 + \beta_3, \quad \beta! = \beta_1!\beta_2!\beta_3!,$$

$$\nabla^\beta = (\partial/\partial x^1)^{\beta_1} (\partial/\partial x^2)^{\beta_2} (\partial/\partial x^3)^{\beta_3}$$

the degree of the vector $(\mathbf{x} - \mathbf{x}_i)^\beta$ is defined analogously. The inner summation in the series (2.11) is a homogeneous harmonic polynomial P_k of degree k in the argument $\mathbf{x} - \mathbf{x}_i$. In accord with Euler's identity for homogeneous functions we have

$$\frac{\partial P_k}{\partial n} = r_i^{-1}(x^\alpha - x_i^\alpha) \frac{\partial P_k}{\partial x^\alpha} = k r_i^{-1} P_k \quad \text{on} \quad \Gamma_i$$

From this, taking into account the fact that two homogeneous harmonic polynomials of differing degrees are orthogonal on the sphere, we obtain

$$A_{ijm}^{k,\alpha} = k r_i^{k-1} \int_{|\mathbf{y}|=1} \sum_{|\beta|=k} \frac{1}{\beta!} \nabla^\beta \Phi_j^m(\mathbf{x}_i, t)\, \mathbf{y}^\beta P_{k,\alpha}(\mathbf{y})\, d\mathbf{y} \tag{2.12}$$

According to Parseval's equality we have

$$|A_{ijm}^k| = k r_i^{k-1} |\nabla^k \Phi_j^m(\mathbf{x}_i, t)|$$

$$|\nabla^k \Phi(\mathbf{x})|^2 \equiv \int_{|\mathbf{y}|=1} \Big| \sum_{|\beta|=1} \frac{1}{\beta!} \nabla^\beta \Phi(\mathbf{x})^\beta \mathbf{y} \Big|^2 d\mathbf{y} \tag{2.13}$$

We square both sides of the equation (2.11) and integrate over the sphere $|\mathbf{x} - \mathbf{x}_i| = R_j$. By virtue of the orthogonality of the terms of the series (2.11), we have

$$R_j^{2k+2} |\nabla^k \Phi_j^m(\mathbf{x}_i, t)|^2 \leqslant \int_{|\mathbf{x} - \mathbf{x}_i| = R_j} |\Phi_j^m|^2 dS \tag{2.14}$$

Since Φ_j^m is a homogeneous function of $\mathbf{x} - \mathbf{x}_j$ of degree $-m-1$, it follows that

$$|\Phi_j^m(\mathbf{x}, t)| \leqslant |\mathbf{x} - \mathbf{x}_j|^{-m-1} \sqrt{\frac{2m+1}{4\pi}} \frac{r_j^{m+2}}{m+1} |w_j^m| \tag{2.15}$$

For the proof of this inequality we need to estimate Φ_j^m in terms of the integral of its square over the sphere $|\mathbf{x}-\mathbf{x}_j| = 1$, which may be expressed in terms of $|w_j^m|$ by virtue of the relation (2.10).

From the inequalities (2.14) and (2.15) it follows that

$$|\nabla^k \Phi_j^m(x_i, t)| \leqslant \sqrt{3/8}\,|w_j^m|\,(r_j^{m+2}/(m+1))\,R_j^{-m-k-1}$$
$$0 < R_j \leqslant |x_j - x_i|/2, \qquad k \geqslant 0, \qquad m \geqslant 0 \tag{2.16}$$

In accordance with the relations (2.9) and (2.13)-(2.16) we have

$$\sum_{m=1}^{\infty} |A_{ijm}^k| \leqslant \sqrt{3/8}\;\; k r_i^{k-1} r_j^3 R_j^{-k-2} \|w_j\|, \qquad R_j \geqslant 2r_j \tag{2.17}$$

Putting $R_j = |\mathbf{x}_j - \mathbf{x}_i|/2$ here, we obtain the desired estimates

$$|w_i^k| \leqslant |f_i^k| + \sqrt{3/8}\max_j \|w_j\|\, k\,(2r/l)^{k+2} S_{k+2} \;\; (k \geqslant 1;\; i = 1, \ldots, N) \tag{2.18}$$

where r is the mean radius of the bubbles, l is the mean distance between neighboring bubbles, and

$$S_m = \max_i \sum_{j \neq i} (l/|x_i - x_j|)^m$$

The quantities $\|w_i\|$ satisfy the inequalities (2.4). We estimate the sums S_m by means of an integral. Since there are on the average l^{-3} bubbles in a unit volume, we have

$$S_m \sim (M/l)^{3-m} \;(m = 1,\; 2),\;\; S_3 \sim \ln(M/l),\;\; S_m \sim 1 \;\; (m \geqslant 4) \tag{2.19}$$

Using the relations (2.13) and (2.16) for m = 0, $w_i^{\circ,\circ} = \sqrt{4\pi} s_i$, we obtain

$$|f_i^k| \leqslant \sqrt{4\pi/3}\,|v_i|\,\delta_{1k} + \sqrt{3\pi/2}\max_j |s_j|\, k\,(2r/l)^{k+1} S_{k+1}$$
$$\|f_i\| \leqslant \sqrt{4\pi/3}\,|v_i| + \text{const}\max_j |s_j|\, r^2 l^{-3} M \tag{2.20}$$

where the const $< \infty$ if $2r/l < 1$; $\delta_{ii} = 1$, $\delta_{ik} = 0$ for $i \neq k$.

We write out the equation for determining $w_i^{1,\alpha}$:

$$w_i^\alpha \equiv \sqrt{3/(4\pi)}\, w_i^{1,\alpha} = v_i^\alpha - \partial\varphi_i'(x_i, t)/\partial x^\alpha \left(\varphi_i' = \sum_{j\neq i}\varphi_j;\;\; \alpha = 1, 2, 3;\;\; i = 1, \ldots, N\right) \tag{2.21}$$

From this and from the relations (2.16) and (2.18), we have

$$w_i = v_i - \nabla(\Phi_i^{\circ\prime} + \Phi_i^{1\prime})\,|_{x_i} + O(sc + vc^{2/3}) \tag{2.22}$$

where s and v are typical values of s_i and $|v_i|$, respectively; $\Phi_i^{m\prime}$ is the sum of Φ_j^m with respect to the index $j \neq i$.

We calculate the limit of φ as $l \to 0$ ($c > 0$, $s/r \neq \infty$). It is necessary to determine the class of allowable distributions of bubbles for which the limit of φ exists and does not depend on a specific distribution from this class. We point out two extreme examples of such classes: 1) arbitrary distributions; in this case no limit exists which is the same for all distributions; 2) a uniform distribution, wherein the bubbles are at the same distance from each other. This class does not contain distributions of bubbles in real flows in which the distance between bubbles varies as flow motion takes place.

We assign a class of allowable distributions by means of the regular functions $r'(\mathbf{x}, t) > 0$, $s'(\mathbf{x}, t)$, $\mathbf{w}(\mathbf{x}, t)$, $c(\mathbf{x}, t)$, defined in a domain G with bubbles:

(1) $r_i = r_0 r'(x_i, t)$, $s_i = r_0 s'(x_i, t)$, $w_i = w(x_i, t)$ $(i = 1, \ldots, N)$

(2) We decompose domain G into cubes g with the same volume L^3, $L = r_0^{a}$, $2/3 < a < 1$. The number of bubbles in the cube g is

$$c(x, t)/(4\pi/3\, r'(x, t)^3)\,(L/r_0)^3,\; x \in g$$

Within the confines of g these bubbles can be distributed arbitrarily; in particular, they may all be grouped in one place.

THEOREM 2.2. Subject to the conditions (1) and (2), we have

$$\max_{\mathbf{x}\in\Omega} |\varphi(\mathbf{x}, t) - \Phi(\mathbf{x}, t)| \leqslant \text{const } r_0^{3a-2}, \quad 2/3 < a < 1 \tag{2.23}$$

where the const depends only on the maxima of the functions r', s', w, and c, on the maxima of their first derivatives with respect to **x**, and on the minimum of r' in G; in addition,

$$\Phi(\mathbf{x}, t) = \frac{3}{4\pi}\int_G c(\mathbf{y}, t)\left(-\frac{s'(\mathbf{y}, t)}{r'(\mathbf{y}, t)} + {}^1/_2\mathbf{w}(\mathbf{y}, t)\nabla\right)|\mathbf{x}-\mathbf{y}|^{-1}\,d\mathbf{y} \tag{2.24}$$

Proof. Let $\Phi^m = \sum_{i=1}^{N} \Phi_i^m$. We show that $\Phi^\circ + \Phi^1$ converges to the integral (2.24) and that $\Phi^m \to 0$ for $r_0 \to 0$ when $m \geq 2$.

To estimate the difference between the integral (2.24) and $\Phi^\circ + \Phi^1$ we decompose the domain G containing the bubbles into cubes g of volume L^3, as stated in the condition (2), so that the point **x** will be the center of some cube g_0. The values of the integral over $g \neq g_0$ and of the sum $\sum_{\mathbf{x}_i\in g} (\Phi_i^\circ + \Phi_i^1)$ are found in a common interval; consequently, the difference of these values does not exceed the length of this interval

$$\text{const } L^3 L \max_{y\in g} (|\mathbf{x}-\mathbf{y}|^{-2} + |\mathbf{x}-y|^{-3}) \tag{2.25}$$

Further

$$\left|\int_{g_0}\right| \leqslant \text{const } L$$

Since

$$|\Phi_i^m| \leqslant \text{const } r_0 \text{ in } \Omega \tag{2.26}$$

it follows that

$$\left|\sum_{\mathbf{x}_i\in g_0} (\Phi_i^\circ + \Phi_i^1)\right| \leqslant \text{const } (L/r_0)^3 r_0$$

From this, summing the expression (2.25) over all $g \neq g_0$, we find that

$$|\Phi - \Phi^\circ - \Phi^1| \leqslant \text{const } (L\ln(M/L) + L + L^3 r_0^{-2}) \tag{2.27}$$

We estimate Φ^m, $m \geq 2$:

$$\left|\sum_{\mathbf{x}_i\in g_0} \Phi_i^m + \sum_{g\neq g_0}\sum_{\mathbf{x}_i\in g} \Phi_i^m\right| \leqslant \text{const } (L^3 r_0^{-2} + \max_i |w_i^m|\, r_0^{m-1} J)$$

$$J = L^3 \sum_{g\neq g_0} \max_{y\in g} |\mathbf{x}-\mathbf{y}|^{-m-1} \leqslant \text{const} \begin{cases} \ln L^{-1}, & m = 2 \\ L^{-m+2}, & m > 2 \end{cases}$$

Since the parameter r_0 does not enter into the estimate (2.18) for $|w_i^m|$, then, substituting here and in the expression (2.27) $L = r_0^a$, $2/3 < a < 1$, we obtain the inequality (2.23).

This completes the proof of the theorem.

The velocity $\nabla\varphi$ does not converge for $l \to 0$ (i.e., $r_0 \to 0$) uniformly with respect to $\mathbf{x} \in \Omega$. We can show merely that it is bounded. Therefore we restrict further the class of distributions of bubbles to be considered. We assume that

(3) for almost all bubbles

$$\nabla\varphi_i'(\mathbf{x}, t) \to \nabla\Phi(\mathbf{x}, t) \quad \text{for } l \to 0, \ \mathbf{x} \in \Omega_i$$

where Ω_i is a ball of radius r_i with center $\mathbf{x}_i$.

THEOREM 2.3. Subject to the conditions (1)-(3) the equation

$$\int ((1-c)\,\partial\psi/\partial t + (\nabla\Phi + {}^1/_2 c\mathbf{w})\nabla\psi)\,d\mathbf{x}\,dt = 0 \tag{2.28}$$

is valid for an arbitrary smooth finite – i.e., equal to zero outside of some ball and beyond a finite time interval – function $\psi(\mathbf{x}, t)$.

Proof. We have the identity

$$\int_{-\infty}^{\infty} \Big(\int_{\Omega} (\partial\psi/\partial t - \varphi\Delta\psi)\,d\mathbf{x} - \int_{\Gamma} \varphi\,\partial\psi/\partial n\,\partial S\Big)\,dt = 0 \tag{2.29}$$

where Γ is the surface of all the bubbles.

We let $r_0 \to 0$.

From inequality (2.23) it follows that

$$\int_{\Omega} (\partial\psi/\partial t - \varphi\Delta\psi)\,d\mathbf{x} \to \int (\partial\psi/\partial t - \Phi\Delta\psi)(1-c)\,d\mathbf{x} \tag{2.30}$$

We transform

$$\int_{\Gamma_i} \varphi \frac{\partial\psi}{\partial n}\,dS = \int_{\Omega_i} (\nabla\varphi_i'\nabla\psi + \varphi_i'\Delta\psi)\,d\mathbf{x} + \int_{\Gamma_i} \Big(-s_i r_i - {}^1/_2 r_i \mathbf{w}_i\cdot\mathbf{n} + \sum_{m=2}^{\infty} \Phi_i^{m}\Big)\frac{\partial\psi}{\partial n}\,dS$$

Expanding ψ by Taylor's formula in a neighborhood of the point $\mathbf{x}_i$ and taking into account the orthogonality properties of the spherical functions, we obtain

$$\int_{\Gamma_i} \varphi \frac{\partial\psi}{\partial n}\,dS = \int_{\Omega_i} (\nabla\varphi_i'\nabla\psi + \varphi_i'\Delta\psi)\,d\mathbf{x} - \frac{4\pi}{3} r_i^3 \mathbf{w}_i \nabla\psi(\mathbf{x}_i, t) + O(r_i^4) \tag{2.31}$$

Next, by virtue of the relations (2.26) and (2.23), we have

$$\varphi_i'(\mathbf{x}, t) \to \Phi(\mathbf{x}, t) \text{ for } r_0 \to 0,\ \mathbf{x} \in \Omega_i$$

From this and from the relations (2.23), (3), and (2.31) it follows that

$$\int_{\Gamma} \varphi \frac{\partial\psi}{\partial n}\,dS \to \int ((\nabla\Phi - {}^1/_2\mathbf{w})\nabla\psi + \Phi\Delta\psi)\,c\,d\mathbf{x} \tag{2.32}$$

Integrating the relations (2.30) and (2.32) with respect to the time t, we obtain, in accord with the equation (2.29), the equation (2.28).

This completes the proof of the theorem.

From the identity (2.28) we obtain the following expression for the mass flow of the mixture:

$$\langle\rho\nabla\varphi\rangle \to \langle\rho\rangle\nabla\Phi + {}^1/_2\,c\mathbf{w} \text{ for } l \to 0 \tag{2.33}$$

3. Equations of Motion for the System of Bubbles

Since the state of the motion of a liquid with bubbles of spherical shape is uniquely determined by the values of the 8N numerical parameters $\mathbf{x}_i$, $\mathbf{v}_i$, r_i, s_i ($i = 1, \ldots, N$), its evolution in time is described by a system of ordinary differential equations of Lagrange (see [9]). In order to write out this system in explicit form for a large number of bubbles we need an additional assumption concerning the smallness of their volume concentration. For the case in which the radii of all the bubbles are the same and are constant in time (i.e., incompressible globules of the same radius are considered), this was done in [10, 11].

The assumption concerning the spherical shape of the bubbles is satisfied approximately. In actuality, the bubbles have a nonspherical shape, unknown beforehand, which it is necessary to determine in the course of solving the problem. Therefore, strictly speaking, the state of motion of a liquid with bubbles cannot be defined by a finite number of parameters. The following theorem is found to be applicable.

THEOREM 3.1. The Lagrange equations of motion are a finite-dimensional Galerkin approximation to the exact problem under which the energy-conservation law remains valid.

Proof. We first formulate the exact problem.

We assume that the pressure $p_i (i = 1, \ldots, N)$ inside the i-th bubble is constant throughout its volume and is given. According to the Cauchy–Lagrange integral, this yields the boundary condition

$$\partial\varphi / \partial t + {}^1/_2 \, |\nabla\varphi|^2 + p_i - \varkappa K + U = 0 \quad \text{on} \quad \Gamma_i (i = 1, \ldots, N) \tag{3.1}$$

where $\varkappa$ is a surface tension constant, K is the curvature of the surface Γ_i, and $K(\mathbf{x}) > 0$ ($\mathbf{x} \in \Gamma_i$) if Γ_i is convex in a neighborhood of the point $\mathbf{x}$. A kinematic condition for nonpermeability of the bubble walls must also be satisfied, namely,

$$\partial\varphi / \partial n = v_n \quad \text{on} \quad \Gamma = \bigcup_{i=1}^{N} \Gamma_i \tag{3.2}$$

where v_n is the normal velocity of the bubble walls. The problem consists in determining the surface Γ and the velocity potential $\varphi(\mathbf{x}, t)$, so that $\varphi(\mathbf{x}, t)$ will be harmonic outside Γ (in Ω), $\varphi|_\infty = 0$, and such that the boundary conditions (3.1) and (3.2) will be satisfied.

For an arbitrary smooth time-dependent surface Γ and for an arbitrary harmonic function $\varphi, \varphi|_\infty = 0$, defined outside of Γ, we have the identity

$$\frac{d}{dt} T \equiv \frac{d}{dt} \frac{1}{2} \int_\Omega |\nabla\varphi|^2 d\mathbf{x} = - \int_\Gamma \left(\frac{\partial\varphi}{\partial t} \frac{\partial\varphi}{\partial n} + \frac{1}{2} |\nabla\varphi|^2 v_n \right) dS \tag{3.3}$$

where T is the kinetic energy of the liquid. If Γ and φ satisfy the boundary conditions (3.1) and (3.2), i.e., if they constitute a solution of the problem, we then obtain from the relation (3.3) the energy-conservation law

$$\frac{d}{dt} T = \sum_{i=1}^{N} \left(p_i \frac{d}{dt} V_i - \varkappa \frac{d}{dt} S_i + \int_{\Gamma_i} U v_n dS \right) \tag{3.4}$$

where S_i and V_i are the surface area and volume of the i-th bubble. We have used here the equations

$$\frac{d}{dt} V_i = \int_{\Gamma_i} v_n \, dS, \qquad \frac{d}{dt} S_i = \int_{\Gamma_i} K dS \tag{3.5}$$

We restrict the class of surfaces Γ and potentials φ considered. In doing this, it is necessary to renounce the exact satisfaction of the boundary conditions; however, the energy-conservation law (3.4) will not be violated.

We assume that the surface Γ is a given (fixed) function of a finite number of parameters $q = (q_1, \ldots, q_M)$ (generalized coordinates) and does not depend on the time. Variation of Γ with time occurs only as the result of a change in q. For fixed q the normal velocity of the surface Γ is a linear function of the generalized velocities q_j'; thus

$$v_n = \sum_{j=1}^{M} \chi_j(\mathbf{x}, q) q_j', \quad \mathbf{x} \in \Gamma, \quad q_j' \equiv \frac{dq_j}{dt} \tag{3.6}$$

If we assume the bubbles are spherical, then for the quantities $q_{4i-3}, q_{4i-2}, q_{4i-1}, q_{4i}$ it is necessary to take the coordinates of the center and the radius of the i-th bubble. In this case $M = 4N$ and

$$(\chi_{4i-3}, \chi_{4i-2}, \chi_{4i-1}, \chi_{4i}) = (\mathbf{n}, 1) \text{ on } \Gamma_i, \ 0 \text{ on } \Gamma_k \ (k \neq 1)$$

We shall satisfy the flow condition (3.2) exactly, i.e., at each point of the surface Γ. Then for the velocity potential we have

$$\varphi(\mathbf{x}, t) = \sum_{j=1}^{M} \psi_j(\mathbf{x}, q) q_j', \quad \Delta\psi_j = 0 \text{ in } \Omega, \quad \psi_j|_\infty = 0, \ \frac{\partial\psi_j}{\partial n} = \chi_j \quad \text{on} \quad \Gamma \tag{3.7}$$

The boundary condition (3.1) cannot be satisfied at each point of the surface Γ. We can satisfy it in the mean in the following sense. We take M linearly independent functions $\chi_1^*(\mathbf{x}, t), \ldots, \chi_M^*(\mathbf{x}, t)$, $\mathbf{x} \in \Gamma$, and we put

$$\sum_{i=1}^{N} \int_{\Gamma_i} (\partial\varphi / \partial t + {}^1/_2 |\nabla\varphi|^2 + p_i - \varkappa K + U) \chi_j^* \, dS = 0 \quad (j = 1, \ldots, M) \tag{3.8}$$

We obtain M equations for determining M unknown functions of the time q(t). This is the essence of the Galerkin method. Under weak restrictions on the functions χ_j^* (the system of functions $\{\chi_j^*\}$ must be complete), the system of equations (3.8) for $M = \infty$ is equivalent to the boundary condition (3.1). For $M < \infty$ we have the finite-dimensional approximation of the problem.

We select the system of functions $\{\chi_j^*\}$, starting from the fact that the energy-conservation law (3.4) is not violated. The equation (3.8) is satisfied for an arbitrary linear combination of the functions $\chi_1^*, \ldots, \chi_M^*$, i.e., for an arbitrary function χ^* from the linear set X with the basis $\{\chi_1^*, \ldots, \chi_M^*\}$. For the energy-conservation law (3.4), subject to the condition (3.2), to be valid, it is necessary and sufficient that the equation (3.8) be satisfied for an arbitrary normal velocity $\chi^* = \partial\varphi/\partial n$ from the class considered. For fixed q the velocities $\partial\varphi/\partial n$ form an M-dimensional linear set (3.7), which must coincide, consequently, with X, i.e.,

$$\chi_j^* = \chi_j \qquad (j = 1, \ldots, M) \tag{3.9}$$

We may confirm that the equations (3.7)-(3.9) are equivalent to the system of Lagrange equations

$$\frac{d}{dt}\frac{\partial T}{\partial q_j'} - \frac{\partial T}{\partial q_j} = \sum_{i=1}^{N} \frac{\partial}{\partial q_j}\Big(p_i V_i - \varkappa S_i + \int_{\Omega_i} U \, d\mathbf{x}\Big) (j = 1, \ldots, M) \tag{3.10}$$

In accordance with equations (3.5), we have

$$\sum_{i=1}^{N} \int_{\Gamma_i} (p_i - \varkappa K + U) \chi_j \, dS = \sum_{i=1}^{N} \frac{\partial}{\partial q_j} \Big(p_i V_i - \varkappa S_i + \int_{\Omega_i} U \, d\mathbf{x} \Big) \tag{3.11}$$

In [9] it was shown that, subject to the conditions (3.7), the identities

$$\int_{\Gamma} \Big(\frac{\partial\varphi}{\partial t} + {}^1/_2 |\nabla\varphi|^2 \Big) \chi_j \, dS = - \frac{d}{dt}\frac{\partial T}{\partial q_j'} + \frac{\partial T}{\partial q_j} \qquad (j = 1, \ldots, M) \tag{3.12}$$

are valid for an arbitrary function q(t).

The Lagrange equations (3.10) follow from the relations (3.11) and (3.12).

This completes the proof of the theorem.

Since it has been established that the Lagrange equations for the motion of N spherical bubbles are a finite-dimensional (4N-dimensional) approximation to the exact problem, various modifications of these equations are then possible for a given dimensionality of approximation 4N. The Lagrange approximation is inconvenient in the sense that the velocity potential must be calculated as a solution of the Neumann problem (3.7); this cannot always be done explicitly. Therefore in the sum (3.7) the harmonic functions $\psi_j(\mathbf{x}, q)$ may be assigned a priori, and new variables q_j^* can be introduced in place of the coefficients q_j'. Then the flow condition (3.2) can be satisfied in the mean; this yields equations for determining q_j^* in terms of q_j'.

We consider next the Lagrange equations (3.10) for bubbles of spherical shape. We choose as the generalized coordinates the coordinates of the centers and the radii of the bubbles. We find an approximate expression for the kinetic enefgy, assuming that the volume concentration of the bubbles $c \ll 1$ and that $s = O(r)$.

We put

$$\varphi = \varphi_i + \varphi_i', \qquad \varphi_i = \sum_{m=0}^{\infty} \Phi_i^m$$

Taking into account the fact that φ_i' is a harmonic function inside Γ_i, we obtain

$$2T = -\int_{\Gamma} \varphi \frac{\partial \varphi}{\partial n} dS = -\sum_{i=1}^{N} \left(\int_{\Gamma_i} \left(\varphi_i \frac{\partial \varphi_i}{\partial n} + \varphi_i \frac{\partial \varphi_i'}{\partial n} + \varphi_i' \frac{\partial \varphi_i}{\partial n} \right) dS + \int_{\Omega_i} |\nabla \varphi_i'|^2 d\mathbf{x} \right) \tag{3.13}$$

In the sum φ_i we discard the spherical harmonics Φ_i^m of order $m \geq 2$. The error δ, which arises in the integral over Γ_i, is now transformed and estimated with the help of the inequalities (2.16) and (2.18):

$$|\delta| \leqslant \sum_{m=2}^{\infty} \frac{r_i^3}{m+1} (|w_i^m|^2 + |w_i^m| \, |\nabla^m \varphi_i'(\mathbf{x}_i, t)| \, r_i^{m-1}) = O(r^3 c^{4/3}) \tag{3.14}$$

Since $\nabla\varphi_i' = O(c)$ on Ω_i, then, making an error $O(c^3)$, we neglect the integrals over Ω_i in the equation (3.13). Then

$$2T = \sum_{i=1}^{N} \left(\frac{2\pi}{3} r_i^3 (|\mathbf{w}_i|^2 - \mathbf{w}_i \cdot \nabla\varphi_i'(\mathbf{x}_i, t)) + 4\pi r_i^2 (r_i s_i^2 - s_i \varphi_i'(\mathbf{x}_i, t)) \right) + O(c^3) \tag{3.15}$$

In accordance with the relation (2.14), we make the substitution $\mathbf{w}_i = \mathbf{v}_i - \nabla\varphi_i'(\mathbf{x}_i, t)$. We obtain an expression for the kinetic energy

$$2T = \sum_{i=1}^{N} \left(\frac{2\pi}{3} r_i^3 (|\mathbf{v}_i|^2 - {}^3/_2 \mathbf{v}_i \cdot \nabla\varphi_i'(x_i, t)) + 4\pi r_i^2 (r_i s_i^2 - s_i \varphi_i'(\mathbf{x}_i, t)) \right) + O(c^3) \tag{3.16}$$

By virtue of the relations (2.22), (2.16), and (2.18) we can put

$$\varphi_i'(\mathbf{x}, t) \approx \sum_{j \neq i} (-r_j^2 s_j + {}^1/_2 r_j^3 \mathbf{v}_j \cdot \nabla) |\mathbf{x} - \mathbf{x}_j|^{-1} \tag{3.17}$$

into the equation (3.16) without decreasing its accuracy.

Then

$$\begin{aligned}
\partial T / \partial \mathbf{v}_i &= (2\pi/3) r_i^3 (\mathbf{v}_i - 3\nabla\varphi_i'(\mathbf{x}_i, t)) \\
\partial T / \partial s_i &= 4\pi r_i^2 (r_i s_i - \varphi_i'(\mathbf{x}_i, t)) \\
\partial T / \partial \mathbf{x}_i &= -2\pi\nabla(r_i^3 \mathbf{v}_i \cdot \nabla\varphi_i'(\mathbf{x}_i, t) + 2r_i^2 s_i \varphi_i'(\mathbf{x}_i, t)) \\
\partial T / \partial r_i &= 4\pi r_i^2 ({}^1/_4 |\mathbf{v}_i|^2 - {}^3/_2 \mathbf{v}_i \cdot \nabla\varphi_i'(\mathbf{x}_i, t) + {}^3/_2 s_i^2 - (2s_i / r_i)\varphi_i'(\mathbf{x}_i, t))
\end{aligned} \tag{3.18}$$

We calculate the right sides of the equations (3.10)

$$(\partial / \partial r_i) \quad (p_i V_i - \varkappa S_i) = 4\pi r_i^2 (p_i - 2\varkappa / r_i) \equiv 4\pi r_i^2 p^+ (r_i) \tag{3.19}$$

The derivatives with respect to all the $\mathbf{x}_j$ and the remaining r_j $(j \neq i)$ are equal to zero. For small bubble radii

$$\frac{\partial}{\partial \mathbf{x}_i} \int_{\Omega_i} U d\mathbf{x} \approx \frac{4\pi}{3} r_i^3 \nabla U(\mathbf{x}_i, t), \quad \frac{\partial}{\partial r_i} \int_{\Omega_i} U d\mathbf{x} \approx 4\pi r_i^2 U(\mathbf{x}_i, t) \tag{3.20}$$

(if $\Delta U = 0$, these equations are then true for arbitrary r_i), the derivatives with respect to $\mathbf{x}_j$, r_j $(j \neq i)$ are equal to zero.

Substituting the expressions (3.18)-(3.20) into the equations (3.10), we obtain the desired system of equations for the motion of the N bubbles:

$$\begin{gathered}
d\mathbf{x}_i / dt = \mathbf{v}_i \\
d\mathbf{v}_i / dt + (3s_i / r_i) \, (\mathbf{v}_i - \nabla\varphi_i'(\mathbf{x}_i, t)) - (3\nabla\partial / \partial t)\varphi_i'(\mathbf{x}_i, t) = 2\nabla U(x_i, t) \\
\partial r_i / dt = s_i \\
r_i \frac{ds_i}{dt} + \frac{3}{2} s_i^2 = \frac{\partial}{\partial t} \varphi_i'(\mathbf{x}_i, t) - \frac{1}{2} \mathbf{v}_i \cdot \nabla\varphi_i'(\mathbf{x}_i, t) + \frac{1}{4} |\mathbf{v}_i|^2 + p^+(r_i) + U(\mathbf{x}_i, t) \qquad (i = 1, \ldots, N)
\end{gathered} \tag{3.21}$$

4. Hydrodynamic Equations

In the expressions (3.17) for the $\varphi_i'(\mathbf{x}, t)$ we replace $\mathbf{v}_i$ by $\mathbf{w}_i$. There is then no decrease in the accuracy of the expression (for $c \ll 1$); we also make the assumptions (1)-(3) of Section 2. Then the limit of $\varphi_i'(\mathbf{x}_i, t)$ as $l \to 0$ coincides with $\Phi(\mathbf{x}_i, t)$, as given by equation (2.24). In the equations (3.21) we replace φ_i' by the value of Φ in a neighborhood of $\mathbf{x}_i$. For this it is necessary to assume that

(4) for almost all bubbles

$$\frac{\partial}{\partial t}\varphi_i'(\mathbf{x}_i, t) \to \frac{\partial}{\partial t}\Phi(\mathbf{x}_i, t),$$

$$\frac{\nabla\partial}{\partial t}\varphi_i'(\mathbf{x}_i, t) \to \frac{\nabla\partial}{\partial t}\Phi(\mathbf{x}_i, t) \quad \text{for} \quad l \to 0$$

According to equation (2.24) the function Φ satisfies the differential equation

$$\nabla((1 - 3c/2)\nabla\Phi + {}^3/_2\, c\mathbf{v}) = 3cs/r \tag{4.1}$$

and the condition

$$\Phi|_\infty = 0$$

From the equations (3.21) we obtain in the limit, making the assumptions (1)-(4), the results

$$d\mathbf{v}/dt + 3s(\mathbf{v} - \nabla\Phi)/r - 3\nabla(\partial\Phi/\partial t + {}^1/_2|\nabla\Phi|^2) = 2\nabla U \tag{4.2}$$

$$\partial\Phi/\partial t + {}^1/_2|\nabla\Phi|^2 + {}^1/_4|\mathbf{v} - \nabla\Phi|^2 + p^+(r) + U - r_0^2 - (r\,ds/dt + {}^3/_2 s^2) = 0 \tag{4.3}$$

$$dr/dt = s \qquad (d/dt = \partial/\partial t + \mathbf{v}\cdot\nabla) \tag{4.4}$$

We have here denoted $r' = r/r_0$, $s' = s/r_0$ by r and s, so that in these expressions r and s are not small; $r_0 \ll M$ is a typical bubble radius. We have also added in here several terms of order c^2 [in equation (4.2) we have added in the term $-{}^3/_2|\nabla\Phi|^2$, and in equation (4.3) the term ${}^3/_4|\nabla\Phi|^2$]. This does not diminish the accepted accuracy of the approximation. This is done so that the equations will be invariant relative to a Galilean transformation (see below). The term of order r_0^2 is left only with the higher derivative since only it is essential as $r_0 \to 0$.

The equation for the conservation of the number of bubbles has the form

$$dc/dt + c\nabla\cdot\mathbf{v} = 3cs/r \tag{4.5}$$

Equations (4.1)-(4.5) constitute a closed system of hydrodynamic equations.

Eliminating s from the equations (4.1) and (4.5), we obtain

$$\partial c/\partial t - \nabla((1 - 3c/2)\nabla\Phi + {}^1/_2 c\mathbf{v}) = 0 \tag{4.6}$$

which is equivalent to the identity (2.28). By virtue of the relation (2.33) the equation (4.6) is the continuity equation.

Putting $r_0 = 0$ in equation (4.2), we obtain the limiting equations. The function $p^+(r)$ must either be constant or depend on $r' = r/r_0$.

We study some general properties of the system of equations (4.1)-(4.5).

Satisfaction of the Inequalities $r > 0$, $c > 0$. In accord with the physical meaning of the variables we must have $r \geq 0$, $c \geq 0$, $c \leq 1$. From equation (4.5) it follows that if $|s/r| < \infty$ and $c \neq 0$ at some point $\mathbf{x}_0$ at time t_0, then $c \neq 0$ along the entire bubble trajectory (i.e., the curve $d\mathbf{x}/dt = \mathbf{v}$) passing through $\mathbf{x}_0$. Next, eliminating s from the equations (4.1) and (4.4), we obtain

$$dr/dt = r\nabla\cdot\mathbf{N}/3c; \quad \mathbf{N} = (1 - 3c/2)\nabla\Phi + {}^3/_2 c\mathbf{v}$$

Therefore, if $|\nabla\cdot\mathbf{N}|/(3c) < \infty$ and $r \neq 0$ at some point, it follows that $r \neq 0$ along the entire bubble trajectory passing through this point. For smooth solutions, on which s/r and $\nabla\cdot\mathbf{N}/(3c)$ are bounded, the inequalities $c > 0$ and $r > 0$ are satisfied for all $t > 0$ if their initial values are positive.

The inequality $c \leq 1$ is not guaranteed if $\mathbf{v} \neq \nabla\Phi$. Bubbles are "allowed" to be displaced relative to the liquid; therefore they pile up, forming a cavity. The hydrodynamic equations are valid outside this cavity.

Galilean Transformation and Dilatation. The equations (4.1) and (4.5) do not change their form when the following change of variables is made:

$$\mathbf{x} = \mathbf{a}t + \mathbf{x}', \quad \Phi = \mathbf{a}\cdot\mathbf{x} - {}^1/_2\,|\,\mathbf{a}\,|^2 t + b + \Phi', \quad \mathbf{v} = \mathbf{a} + \mathbf{v}' \tag{4.7}$$

the remaining variables t, s, r, and c do not change, where **a** and b are arbitrary constants. This change of variables is referred to as a Galilean transformation; physically it corresponds to changing over to a system of coordinates moving relative to the initial system with the constant velocity **a**.

When $p^+(r) \equiv$ const the equations (4.1)-(4.5) are invariant relative to the group of dilatations

$$\mathbf{x} = a\mathbf{x}', \quad t = at', \quad \Phi = a\Phi', \quad r = ar' \tag{4.8}$$

the variables **v**, s, and c do not change; a is an arbitrary constant.

Type of System. The system of equations (4.1)-(4.5) is quasilinear. The equation of its characteristics $\theta(\mathbf{x}, t) = 0$ has the form (for $r_0 > 0$)

$$\begin{gathered} r\mu^2\,|\,\nabla\theta\,|^2\,((1 + 3c)\mu^2 - 9c\mathbf{w}\cdot\nabla\theta\mu + {}^9/_2 c\,(\mathbf{w}\cdot\nabla\theta)^2) = 0 \\ \mu = \partial\theta / \partial t + \mathbf{v}\cdot\nabla\theta, \qquad \mathbf{w} = \mathbf{v} - \nabla\Phi \end{gathered} \tag{4.9}$$

The quadratic trinomial in μ in the parentheses has a negative discriminant if $\mathbf{w} \cdot \nabla\theta \neq 0$ and

$$0 < c < {}^2/_3 \tag{4.10}$$

and, consequently, is not equal to zero. Therefore, subject to the condition (4.10) and $r > 0$, we have the real characteristics

$$\partial\theta / \partial t + \mathbf{v}\cdot\nabla\theta = 0 \qquad \text{(multiplicity 2)} \tag{4.11}$$

$$|\,\nabla\theta\,| = 0 \qquad \text{(multiplicity 2)} \tag{4.12}$$

and two distinct complex characteristics for $\mathbf{w} \neq 0$, which become real for $\mathbf{w} = 0$ and merge with the characteristic (4.11).

The system of equations (4.1)-(4.5) is of mixed type. The manifold $\mathbf{w} = 0$ is singular. We note that **w** is the displacement velocity of the bubbles relative to the liquid.

We transform the limiting equations for $r_0 = 0$ in the one-dimensional case: we differentiate equation (4.3) with respect to x and eliminate s with the aid of equation (4.1). The condition of hyperbolicity of the resulting system has the form

$$-{}^3/_2 < u^{-2} r\,dp^+(r) / dr < \beta_0(c) \tag{4.13}$$

The function $\beta_0(c)$ is defined in Section 5.

5. Velocity of Propagation of Small Perturbations

Consider a constant flow in the absence of external forces ($U = 0$)

$$\begin{gathered} \Phi = \mathbf{u}_0\cdot\mathbf{x}, \quad \mathbf{v} = 0, \quad s = 0, \quad r = 1, \quad c = c_0 \\ (p^+(1) = -{}^3/_4\,|\,\mathbf{u}_0\,|^2) \end{gathered} \tag{5.1}$$

The more general case in which $\mathbf{v} = \mathbf{v}_0$ reduces to the equations (5.1) by means of the Galilean transformation (4.7). We linearize the system of equations (4.1)-(4.5) in a neighborhood of the solution (5.1). Let us put

$$\begin{gathered} \Phi = \mathbf{u}_0\cdot\mathbf{x} + \Phi', \quad r = 1 + r', \quad c = c_0\,(1 + c') \\ p^+(r) = p^+(1) - br' \end{gathered}$$

We obtain (we omit the prime)

$$(A - {}^3/_2)\Delta\Phi + {}^1/_2\nabla\cdot\mathbf{v} - s - {}^1/_2\mathbf{u}_0\cdot\nabla c = 0$$
$$-3\nabla(\partial\Phi / dt + \mathbf{u}_0\cdot\nabla\Phi) + \partial\mathbf{v} / \partial t - 3\mathbf{u}_0 s = 0$$
$$\partial\Phi / \partial t + {}^3/_2\mathbf{u}_0\cdot\nabla\Phi - {}^1/_2\mathbf{u}_0\cdot\mathbf{v} - r_0^2\partial s / \partial t - br = 0 \tag{5.2}$$
$$-s + r_t = 0,\ \nabla\cdot\mathbf{v} - 3s + \partial c / \partial t = 0$$
$$(A = 1 / (3c_0) + 1)$$

We consider sinusoidal waves advancing along the x^1 axis

$$\begin{pmatrix} \Phi \\ \mathbf{v} \\ s \\ r \\ c \end{pmatrix} = \mathrm{Re}\begin{pmatrix} \Phi_1 / i \\ \mathbf{v}_1 \\ is_1 \\ r_1 \\ c_1 \end{pmatrix} e^{i(kx^1+\omega t)} \tag{5.3}$$

Substituting this expression into equations (5.2), we obtain a system of homogeneous linear algebraic equations for $\Phi_1, \ldots, c_1$. In order for nonzero solutions of the form (5.3) to exist, the determinant of this system must vanish:

$$Q(k, \omega) = 0$$

Next, we consider two particular cases.

1. $u_0^1 = 0$, i.e., the velocity of the perturbation wave is directed normal to the velocity of the main flow (5.1). Calculations show that

$$Q = \omega^4(\omega^2 + Ak^2(r_0^2\omega^2 - b - {}^3/_2 \mid \mathbf{u}_0 \mid ^2)) = 0 \tag{5.4}$$

From this we obtain the phase velocity of the perturbation wave

$$\lambda = \omega / k = (A(b + {}^3/_2 \mid \mathbf{u}_0 \mid ^2 - r_0^2\omega^2))^{1/2} \tag{5.5}$$

When $\mathbf{u}_0 = 0$ this expression coincides with that given for the model in [5].

All the roots ω of the equation (5.4) are real for arbitrary (real) k. Consequently, the flow (5.1) is stable to such perturbations.

2. $u_0^2 = u_0^3 = 0$, i.e., the perturbation wave propagation velocity is parallel to the velocity of the main flow (5.1). Let $u_0^1 = u_0$. Calculations show that

$$Q = \omega^2(\omega^4 - (3A / 2 + 3)\mathbf{u}_0^2k^2\omega^2 - {}^9/_2\ \mathbf{u}_0^3k^3\omega - (b - r_0^2\omega^2)(Ak^2\omega^2 + 3u_0k^3\omega + {}^3/_2u_0^2k^4)) \tag{5.6}$$

This is a sixth-degree polynomial in ω. It has a zero root of multiplicity two; the other four of its roots are obtained by equating the second factor to zero; thus

$$(\lambda^2 - 3u_0^2 / 2)^2 - (b + 3u_0^2/2 - r_0^2\omega^2)(A\lambda^2 + 3u_0\lambda + {}^3/_2u_0^2) = 0 \tag{5.7}$$

If the pair of numbers λ and u_0 are roots of the equation (5.7), then $-\lambda$ and $-u_0$ will also be roots. Therefore it is sufficient to consider the case $u_0 \geq 0$.

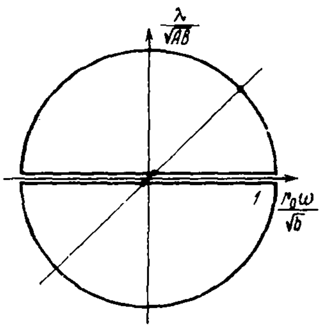

Fig. 2

For $u_0 = 0$ the graph of the function $\lambda = \lambda(\omega)$ is shown in Fig. 2. From this graph we can find the frequency ω as a function of k. It is evident from this figure that for arbitrary k there are four real roots (two of them equal to zero).

Let $u_0 > 0$. From equation (5.7) we express ω in terms of λ. When $0 < c_0 < {}^2/_3$, $u_0 \neq 0$, we have the inequality $A\lambda^2 + 3u_0\lambda + 3u_0^2/2 > 0$ for all λ. Therefore for such values of c_0 we have four distinct cases, depending on the parameter $\beta = bu_0^{-2}$.

1. $\infty > \beta \geq \beta_1(c_0)$. For all k there are two real (and, consequently, two complex-conjugate) roots ω (Fig. 3).

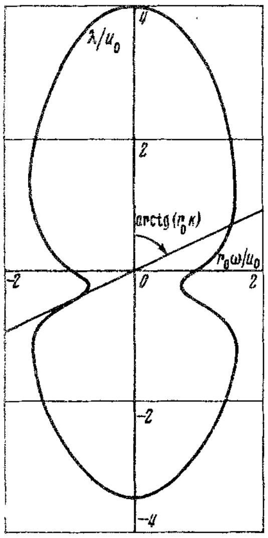

Fig. 3

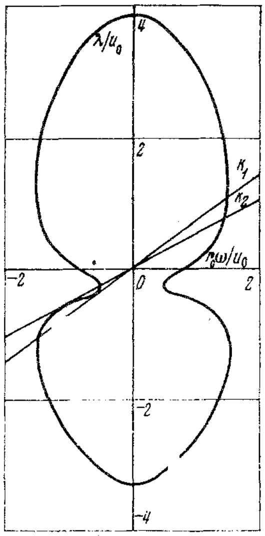

Fig. 4

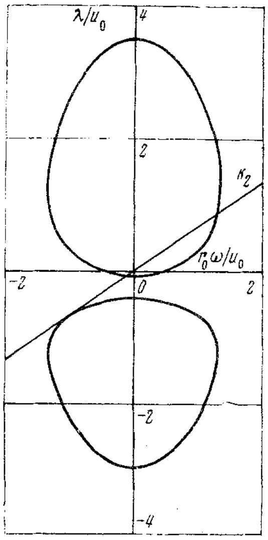

Fig. 5

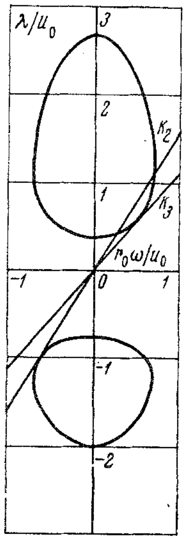

Fig. 6

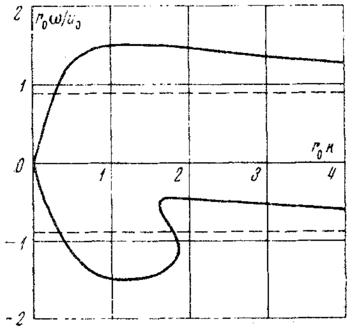

Fig. 7

2. $\beta_1(c_0) > \beta \geq \beta_0(c_0) \approx 3/(1+3c_0)$ (for $c_0 \ll 1$). If $k_1 \leq |k| \leq k_2$, we then have four real roots; if $|k| \notin [k_1, k_2]$, we then have only two real roots (Fig. 4).

3. $\beta_0(c_0) > \beta \geq 0$. For $|k| \leq k_2$ we have four real roots ω, and for $|k| > k_2$ we have only two real roots (Fig. 5).

4. $0 > \beta \geq -3/2$. This case is possible owing to surface tension forces in the bubbles. For $|k| \leq k_2$ we have four real roots ω, for $k_2 < |k| \leq k_3$ we have only two real roots, and for $|k| > k_3$ there are no real roots (Fig. 6).

For $\beta < -3/2$ there are no real roots ω.

Figures 3-6 display the graphs of $\lambda(\omega)$ for $c_0 = 0.1$ and $\beta = 1$, 0.75, 0.25, and -1, respectively.

Figure 7 shows the graph of $\omega(k)$ for $k \geq 0$ for the values $c_0 = 0.1$ and $\beta = 0.75$. For $k < 0$, $\omega(k) = -\omega(-k)$.

Thus, when $r_0 > 0$, the flow (5.1) is unstable relative to short-wave perturbations; when $\beta > \beta_0(c_0)$, it is also unstable relative to long-wave perturbations. When $r_0 = 0$, the flow (5.1) is stable if $-3/2 \leq \beta \leq \beta_0(c)$ and unstable if $\beta > \beta_0$ or $\beta < -3/2$. The hydrodynamic equations (4.1)-(4.5) were introduced under the assumption that the wave length $\gg r_0$. It is therefore necessary to keep this restriction in mind when making a statement concerning instability of the flow in the presence of short-wave perturbations.

6. One-Dimensional Borzhom Problem

We solve the Borzhom problem, formulated in Section 1, for the one-dimensional case in which the bubble velocity is parallel to the $x^1 = x$ axis and all quantities depend only on x and t. Assume that for $x < X(t)$ we have a "pure" stationary liquid with a constant pressure p_-, so that the velocity potential in it is equal to $-p_-t$; let X(t) be the boundary separating the pure liquid and the mixture, which is to be determined. We assume that for $x = \infty$ we have a constant flow, the pressure in which is equal to zero:

$$\Phi_t \to 0,\ \Phi_x \to u_0,\ v \to v_0,\ r \to 1,$$
$$c \to c_0 \text{ for } x \to +\infty$$

where subscripts indicate corresponding derivatives.

Motion occurs as the result of an initial velocity and a drop in pressure. We assume, further, that $p^+(r) \equiv \text{const}$. We consider the limiting equations for $r_0 = 0$. Then from the system (4.1)-(4.5) we easily obtain a closed system of equations for the functions v, $w = v - \varphi_x$, c. To obtain this system we differentiate equation (4.3) (with $r_0 = 0$) with respect to x. The final form of the system is found to be

$$\begin{pmatrix} v \\ w \\ c \end{pmatrix}_t + B\begin{pmatrix} v \\ w \\ c \end{pmatrix}_x = 0, \quad B = \begin{pmatrix} v + w/c & (3 - 1/c)w & 3w^2/(2c) \\ (1 + 1/c)w & v + (^3/_2 - 1/c)w & 3w^2/(2c) \\ -1 + c & 1 - 3c/2 & v - 3w/2 \end{pmatrix} \tag{6.1}$$

We write initial conditions for $t = 0$, $x > X$ and boundary conditions for $t > 0$, $x = X$:

$$v = v_0, \ w = w_0 \ (= v_0 - u_0), \ c = c_0, \ X = 0 \tag{6.2}$$

For $l > 0$ the function $\varphi_i'(x, t)$ is a continuous function of x. According to Theorem 2.2 the potential Φ is the limit of φ_i' as $l \to 0$, uniformly with respect to x. Consequently, the potential Φ must be continuous on the separation boundary:

$$\Phi(X + 0, t) = -p_- t$$

We differentiate this equation with respect to t and substitute the value of Φ_t into equation (4.3) (with $r_0 = 0$); thus

$$\begin{gathered} -X'(v - w) + {}^1/_2(v - w)^2 + {}^1/_4 w^2 = p_- - p^+ \equiv p_1 \\ (X' = (d/dt)X, \ p^+ = -{}^1/_2(v_0 - w_0)^2 - {}^1/_4 w_0^2) \end{gathered} \tag{6.3}$$

Next, we obtain from the identity (2.28) the mass-conservation equation

$$(1 - 3c/2)(v - w) + {}^1/_2 cv = -cX' \tag{6.4}$$

We assume that bubbles neither are formed nor vanish at the separation boundary; therefore

$$X' = v \tag{6.5}$$

We consider a self-similar solution. The problem (6.1)-(6.5) for the determination of the unknowns v, w, c, X admits the group of dilatations (4.8); therefore we seek its solution in the form

$$v = v(\lambda), \ w = w(\lambda), \ c = c(\lambda), \ \lambda = x/t, \ X = Dt \tag{6.6}$$

For the functions v, w, and c we obtain a system of ordinary differential equations

$$(B - \lambda)\frac{d}{d\lambda}\begin{pmatrix} v \\ w \\ c \end{pmatrix} = 0 \quad \text{for} \quad \lambda > D \tag{6.7}$$

The initial conditions (6.2) imply that

$$v \to v_0, \ w \to w_0, \ c \to c_0 \ \text{for} \ \lambda \to \infty \tag{6.8}$$

The boundary conditions (6.3)-(6.5) for $\lambda = D$ assume the form

$$v = (1 - 3c/2)w, \ w = \pm 2(p_1/(1 + 6c - {}^9/_2 c^2))^{1/2}, \ D = v \tag{6.9}$$

Since

$$1 + 6c - {}^9/_2 c^2 > 0 \ \text{for} \ 0 < c < {}^2/_3$$

then for existence of a solution it is necessary that

$$p_1 = p_- + {}^1/_2(v_0 - w_0)^2 + {}^1/_4 w_0^2 > 0 \tag{6.10}$$

The solution of the differential equation (6.7) either has the form

$$\begin{pmatrix} v \\ w \\ c \end{pmatrix} = \text{const} \tag{6.11}$$

or we must have

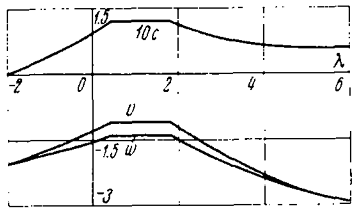

Fig. 8

$$\det(B - \lambda) = (\lambda - v)^3 - {}^1/_2 w^2 (9 + c^{-1})\ (\lambda - v) - {}^9/_2\, w^3 = 0 \quad (6.12)$$

For

$$w \neq 0, \quad 0 < c < c_* = (9(\sqrt[3]{{}^3/_2} - 1))^{-1} \approx 0.74$$

the equation (6.12) has three real distinct roots (eigenvalues of the matrix B)

$$\lambda_i = v + wk_i\,(c)\ (i = 1,\ 2,\ 3) \quad (6.13)$$

where $k_i(c)$ are the roots of the equation

$$k^3 - ({}^9/_2 + {}^1/_2 c^{-1})k - {}^9/_2 = 0$$

(they depend only on c). For $0 < c < c_*$

$$k_1 < k_* = -({}^3/_2)^{2/3} < k_2 < 0,\ 0 < k_3$$

To the eigenvalue λ_i there corresponds the eigenvector of the matrix B

$$\mathrm{r}_i = \begin{pmatrix} k_i + {}^3/_2 \\ k_i + 1 \\ (-k_i^2 + k_i + 3)\,c\,/\,(wk_i) \end{pmatrix} \quad (6.14)$$

Along with the constant solution (6.11) the system of equations (6.7) has three families of solutions corresponding to the three eigenvalues λ_i of the matrix B; thus

$$dv\,/\,r_i^1 = dw\,/\,r_i^2 = dc\,/\,r_i^3, \quad \lambda = \lambda_i\,(v,\ w,\ c) \quad (6.15)$$

The solutions (6.11) and (6.15) are conveniently represented graphically in the phase space of the dependent variables v, w, and c; to the solution (6.11) there corresponds a point; to the solutions (6.15) there correspond three families of nonintersecting (i.e., nonintersecting within the family) curves. To the general solution of the system (6.7) there corresponds a continuous curve, consisting of portions of the curves (6.15) along which the parameter λ is nondecreasing (does not undergo a discontinuity).

We consider the boundary conditions (6.9). The first two of them define a curve K in the phase space of the dependent variables v, w, c. This curve must be joined to the point (v_0, w_0, c_0) by a solution of the system (6.7); in addition, we must have $D \leq \lambda$ on K. Hence, from the roots (6.13) and the third of the conditions (6.9) we find that when $w_0 > 0$ only the third family of the families (6.15) can be adjoined to the curve K; the curves of the first and second families, on which the values of λ are less, cannot be joined to K. The solution of the problem is unique; however it does not exist for all v_0, w_0, c_0, p_1. The set of points (v_0, w_0, c_0) for which the solution exists fills out some set on a surface which depends on p_1. Similarly, when $w_0 < 0$, only solutions of the first or second families of (6.15) can be adjoined to the curve K. The solution is unique; it exists if the point (v_0, w_0, c_0) (for fixed p_1) lies inside some region.

The reason for nonexistence of the solution can be due to one of the following: 1) the conditions on the separation boundary cannot be satisfied; 2) the conditions at infinity (in particular, the condition $w|_\infty = 0$) cannot be satisfied; 3) the condition $c \leq 1$ is violated. In the first case, allowance must be made for the creation and extinction of bubbles on the separation boundary. In the second case it is necessary to consider the system of equations with $r_0 > 0$ (in this case there is also a self-similar solution). In the third case it is necessary to take cavitation into account.

In Fig. 8 we present graphs of the solution of the problem for the initial conditions $v_0 = w_0 = -2.89$, $c_0 = 0.058$ and $p_- = -1.08$ ($p_1 = 1$).

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